Coupled Strip Transmission Lines with Rectangular Inner Conductors*

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Summary—A method is presented for determining the capacitance of electrostatic fields which have hitherto proved intractable because their solutions required the evaluation of hyperelliptic integrals. The method is illustrated by applying it to the determination of the characteristic impedance of a strip transmission line. The results compare favorably with the results of existing solutions. The method is then used to determine the characteristic impedance of coupled strip transmission lines with inner conductors of rectangular shape. Curves are included which permit the determination of this impedance over a wide range of line proportions.

Introduction

THE OBJECTIVES of this paper are: 1) to present a method for determining the capacitance of electric fields which have hitherto proved intractable because their solutions, using conformal mapping techniques, required evaluation of hyperelliptic integrals, and 2) through use of this method, to determine the characteristic impedance of the coupled strip lines of Fig. 1. In the method to be described, the field is divided into two or more simpler configurations, and ordinary conformal mapping techniques are used to find potential distributions in each region. These distributions serve to furnish a first approximation to the correct capacitance and also provide the curvilinear coordinate systems necessary for obtaining a second approximation. Second approximations to the correct potential functions are obtained by using the first terms in Fourier series involving the curvilinear coordinates. Parameters associated with these terms are adjusted so that the capacitance obtained is a maximum. As a result, the capacitance and impedance values are correct to within one or two per cent over the range of proportions investigated. The method described can also be applied to determine the permeance of magnetic fields, the conductance of electric current fields, and the conductance of thermal fields with similarly complicated boundary shapes.

In order to present the method clearly, treatment of the coupled strip lines is preceded by application of the method to the simpler case of the strip transmission line of Fig. 2. The capacitance and characteristic impedance of such a line has been accurately determined,[1] and therefore a check on the accuracy of the method is afforded. After this problem has been treated, the method is applied to determine the characteristic impedance of the coupled strip transmission lines.

* Manuscript received by the PGMTT, June 1, 1956.

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[1] N. A. Begovich, "Capacity and characteristic impedance of strip transmission lines with rectangular inner conductors," IRE Trans., vol. MTT-3, pp. 127–133; March, 1955.

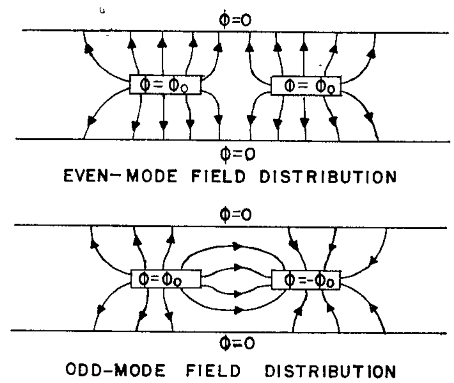

Fig. 1—Field distributions of the even and odd modes in coupled strip line.

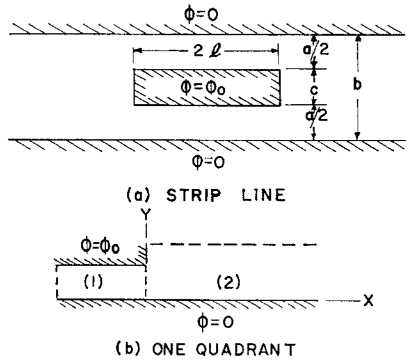

Fig. 2—Strip line configuration.

To determine the capacitance per unit length of the strip transmission line of Fig. 2, an insulating partition is visualized as dividing the quadrant of Fig. 2(b) into the regions 1 and 2. Potential functions are obtained for each region, using conformal transformations as necessary. These functions are the first approximations to the true potential distributions. To obtain the true distributions, an infinite Fourier series could be added in each region, with coefficients evaluated so that all boundary conditions, including those at the surface common to both regions, would be satisfied. However, there are difficulties involved in evaluating these coefficients. Such difficulties may be avoided by seeking a second approximation in which only the first term of the series in each region is retained. The approximation is made to yield an accurate value for the capacitance by: 1) choosing a coordinate system (Fig. 3) in such a way that individual terms in the series exactly satisfy all boundary conditions except those at the matching surface, and 2) determining the coefficients in the series terms in a manner which yields maximum capacitance.

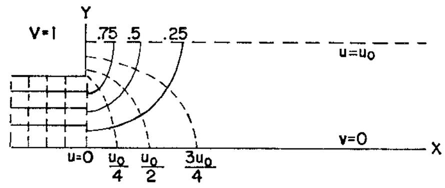

Fig. 3—Strip line coordinate system.

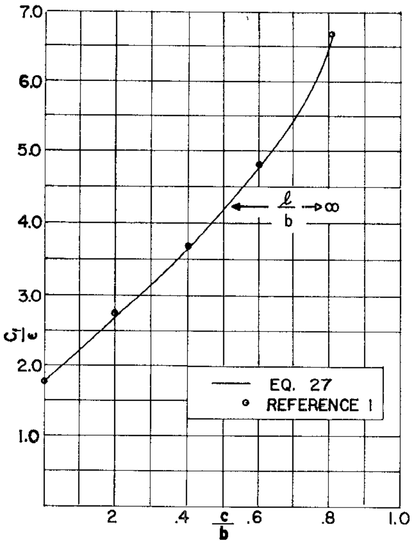

Fig. 4—Fringing capacitance for strip line.

The results of such analysis, shown in Fig. 4 in comparison with the results of Begovich,[1] are accurate to within two per cent of the fringe capacitance. The error, expressed as a percentage of total capacitance, is less than two per cent.

Strip Transmission Line

The problem to be discussed in this section is that of determining the characteristic impedance of the strip transmission line shown in Fig. 2(a). Such a line can propagate a transverse electromagnetic wave (TEM) for which the characteristic impedance, Z_0, is[2]

$$Z_0 = \frac{\sqrt{\mu\epsilon}}{C} = \frac{120\pi}{\sqrt{\epsilon_r}} \cdot \frac{\epsilon}{C}, \tag{1}$$

where ϵ_r is the relative dielectric constant of the material in the space between the electrodes, ϵ is the permittivity, and C is the capacitance per unit length. Thus, the crux of the problem is the determination of the capacitance. The general plan of attack is to determine the potential distribution, the field intensity, the stored energy, and from this, to find the relative capacitance.

In order to determine the potential distribution, an insulating partition is visualized as dividing the field quadrant into the regions 1 and 2 as shown in Fig. 2(b). In region 1 the potential function which satisfies Laplace's equation and the boundary conditions is

$$\phi_1(x, y) = 2y/a, \tag{2}$$

where ϕ_0 in Fig. 2 is taken as unity without loss of generality. In region 2 it is convenient to express the potential in terms of a new system of curvilinear coordinates (u, v), selected so that

$$\phi_2(u, v) = v. \tag{3}$$

Since equipotential surfaces coincide with surfaces of constant value of the coordinate v, the coordinate system (u, v) must be conformally related to the (x, y) system and must satisfy the boundary conditions indicated in Fig. 3 and stated below:

$$\begin{array}{lll} v = 0; & y = 0; & 0 < x \\ v = 1; & x = 0; & a/2 < y < b/2 \\ u = 0; & x = 0; & 0 < y < a/2 \\ u = u_0; & y = b/2; & 0 < x. \end{array} \tag{4}$$

The important relations between (x, y) and (u, v) are developed in the appendix, using the Schwarz-Christoffel transformation. It is shown there that

$$u_0 = K(k)/K(k') \tag{5}$$

$$k = \cos(\pi a/2b) \tag{6}$$

$$k' = \sin(\pi a/2b), \tag{7}$$

where $K(k)$ and $K(k')$ are complete elliptic integrals of the first kind. Consideration of the above leads to a first approximation for the capacitance of Fig. 2(a):

$$C/\epsilon = 8l/a + 4K(k)/K(k'). \tag{8}$$

To obtain a more accurate solution, the insulating partition is removed and the approximate potential solutions are augmented as follows:

$$\phi_1(x, y) = 2y/a + \sum_{m=1}^{\infty} a_m e^{m2\pi x/a} \sin m2\pi y/a \tag{9}$$

$$\phi_2(u, v) = v + \sum_{n=1}^{\infty} b_n e^{-n\pi u} \sin n\pi v. \tag{10}$$

For simplicity, (9) is written for the case of l/a approaching infinity, although it is easily generalized by including a negative exponential term.

Each individual term in the two series satisfies all boundary conditions except those at the matching surface, $u=0$. One method of obtaining the exact potential solutions involves evaluating the coefficients a_m and b_n so that both the potential and the normal component of displacement are continuous across the matching surface. The difficulties associated with this method are great, because the series involve trigonometric functions of two different arguments.

Another method, the one to be pursued here, involves determining the coefficients a_m and b_n in such a way that the potential is continuous across the matching surface,

[2] S. Ramo and J. R. Whinnery, "Fields and Waves in Modern Radio," John Wiley and Sons, Inc., New York, N.Y.; 1944.

and the capacitance attains its maximum value.[3] To pursue this method, the capacitance is expressed as

$$C = 2W/\phi_0^2, \tag{11}$$

where W is the total energy stored in the field and ϕ_0 is the total potential difference across the conductors. In turn, W is expressed as

$$W = (\epsilon/2) \iiint E^2 dV, \tag{12}$$

where E is the magnitude of the field intensity and V is the volume of the field. Thus, assuming the potential difference as unity, capacitance as a function of potential distributions becomes

$$C/\epsilon = 4 \int_0^{a/2} \int_{-1}^{0} [(\partial\phi_1/\partial x)^2 + (\partial\phi_1/\partial y)^2] dx dy + 4 \int_0^1 \int_0^{u_0} [(\partial\phi_2/\partial u)^2 + (\partial\phi_2/\partial v)^2] du dv. \tag{13}$$

Now if the potentials in (9) and (10) are substituted in (13) the result is, for the case of $1/a$ approaching infinity, the expression relating capacitance to the undetermined coefficients, a_m and b_n.

$$C/\epsilon = 81/a + 4u_0 + 2\pi \sum_m m a_m^2 + 2\pi \sum_n (1 - e^{-2n\pi u_0}) n b_n^2. \tag{14}$$

The maximum value of this capacitance is desired, subject to the condition of continuity of the potential $\phi(0, v)$ at the matching surface as expressed by

$$\phi(0, v) = 2y/a + \sum_m a_m \sin m2\pi y/a = v + \sum_n b_n \sin n\pi v. \tag{15}$$

To introduce the potential $\phi(0, v)$ into the capacitance (14), a_m and b_n are expressed in terms of this potential by multiplying (15) by the appropriate sin function, and integrating. When these results are substituted in (14), capacitance is expressed in terms of the potential at the matching surface:

$$C/\epsilon = 81/a + 4u_0 + 2\pi \sum_m m \left\{ (4/a) \int_0^{a/2} [\phi(0, v) - 2y/a] \sin m2\pi y/a\, dy \right\}^2 + 2\pi \sum_n n(1 - e^{-n2\pi u_0}) \left\{ 2 \int_0^1 [\phi(0, v) - v] \sin n\pi v dv \right\}^2. \tag{16}$$

The specific problem at hand now is this: find the function $\phi(0, v)$ which will result in the maximum value of the capacitance. This can be solved approximately by assuming that the potential at the matching surface can be represented by the first term in either of the series in (15). That is, assume

$$\phi(0, v) \approx 2y/a + a_1 \sin 2\pi y/a \approx v + b_1 \sin v. \tag{17}$$

Then, using (17) and the relation between a_m and $\phi(0, v)$ as obtained from (15), a_1 can be expressed in terms of b_1.

$$a_1 = B_0 + B_1 b_1 \tag{18}$$

$$B_0 = (4/a) \int_0^{a/2} (v - 2y/a) \sin 2\pi y/a\, dy \tag{19}$$

$$B_1 = (4/a) \int_0^{a/2} (\sin \pi v) \sin 2\pi y/a\, dy. \tag{20}$$

As shown in the appendix, along the matching surface, y and v are related as follows:

$$v = 1 - F(\theta', k')/K(k') \tag{21}$$

$$\cos \theta' = (k/k') \tan \pi y/b. \tag{22}$$

In (21), $F(\theta', k')$ is the incomplete elliptic integral of the first kind.

Eqs. (18)–(22) serve to uniquely determine a_1 in terms of b_1. Eqs. (19) and (20) are integrated, using numerical or graphical methods.

It is now possible to express the capacitance in terms of the single parameter, b_1:

$$C/\epsilon = 81/a + 4u_0 + 2\pi[(B_0 + B_1 b_1)^2 + (1 - e^{-2\pi u_0}) b_1^2]. \tag{23}$$

When this is maximized with respect to b_1, the result is

$$b_1 = \frac{-B_0 B_1}{1 - e^{-2\pi u_0} + B_1^2}. \tag{24}$$

If $u_0 \gg 1/2\pi$, this reduces to

$$b_1 = \frac{-B_0}{B_1 + \dfrac{1}{B_1}}. \tag{25}$$

When (24) or (25) is substituted in (23), the capacitance is determined. In order to compare the results of this method with results obtained previously,[1] the capacitance is written in the form

$$C/\epsilon = 81/a + C_f/\epsilon, \tag{26}$$

where C_f is the fringe capacitance, given by

$$C_f/\epsilon = 4u_0 + 2\pi[(B_0 + B_1 b_1)^2 + b_1^2]. \tag{27}$$

In Fig. 4, this fringe capacitance is plotted as a function of the ratio c/b for the case of $1/b$ approaching infinity. The results of Begovich[1] are also shown for comparison. A maximum deviation of about two per cent of the fringe capacitance is indicated. It should be noted that this deviation approaches zero as $1/b$ ap-

[3] J. W. S Rayleigh, "The Theory of Sound," Macmillan and Co., Ltd., London, Eng., vol. 2, p. 175; 1896.

proaches zero. Also, when this deviation is expressed as a percentage of total capacitance, the figure is less than two per cent.

Coupled Strip Line: Odd Mode

In this section, the following problem is taken up: for the coupled strip line operating in the odd mode, determine the characteristic impedance. As indicated in Fig. 1, the odd mode is excited by maintaining the inner conductors at equal and opposite potentials with respect to the parallel ground planes. For transverse electromagnetic (TEM) wave propagation on such a line, the characteristic impedance, measured from one strip to ground, can be determined from (1), if the capacitance is taken as that of one strip to ground. As in the previous example, the crux of the problem is the determination of this capacitance. Because of symmetry, attention can be focused on the problem of determining the capacitance of the quadrant shown in Fig. 5.

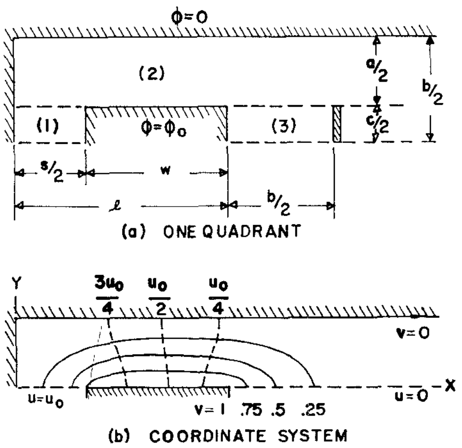

Fig. 5—Quadrant of coupled line in odd mode.

Following the general scheme of the previous section, insulating partitions are assumed to divide the field into the regions marked 1, 2, and 3. In region 1, the potential which satisfies Laplace's equation and the boundary conditions indicated in Fig. 5 is

$$\phi_1(x, y) = 2x/s, \tag{28}$$

where ϕ_0 is taken as unity without loss of generality. In region 2 the solution is

$$\phi_2(u, v) = v, \tag{29}$$

where the coordinate system (u, v) is that sketched in Fig. 5. It is conformally related to the coordinate system (x, y) and satisfies the boundary conditions indicated in Fig. 5. The important relations between (x, y) and (u, v) are developed in the appendix, using the Schwarz-Christoffel transformation. It is shown there that

$$u_0 = 2K(k_0)/K(k_0') \tag{30}$$

$$k_0 = (\tanh \pi w/2a)(\coth \pi(w + s)/2a) \tag{31}$$

$$k_0' = \sqrt{1 - k_0^2}. \tag{32}$$

In region 3 it is impossible to write down a potential solution with the partition in place because the field extends infinitely to the right. Practically, this situation can be remedied by imagining a conducting plate at zero potential to exist at some distance to the right of the conductor as indicated in Fig. 5. In effect, the assumption is made that the energy stored in that portion of region 3 to the right of this conducting plate is negligibly small compared with the energy stored in the entire field. The placement of this plate is somewhat arbitrary. The choice here is a location $b/2$ units to the right of the conductor. Then, in region 3 a suitable potential is

$$\phi_3(x, y) = 1 - 2(x - 1)/b. \tag{33}$$

Corresponding to the above potentials, the first approximation for the capacitance of one strip to ground, operating in the odd mode, is

$$C_0/\epsilon = 4K(k_0)/K(k_0') + 2c/s + 2c/b. \tag{34}$$

To obtain a more accurate solution, the insulating partitions are removed and the approximate solutions for potential are augmented as follows:

$$\phi_1(x, y) = 2x/s + \sum_{m=1}^{\infty} a_m(e^{m2\pi y/s} + e^{-m2\pi c/s}e^{-m2\pi y/s}) \sin m2\pi x/s \tag{35}$$

$$\phi_2(u, v) = v + \sum_{n=1}^{\infty} (b_n e^{-n\pi(u_0-u)} + d_n e^{-n\pi u}) \sin n\pi v \tag{36}$$

$$\phi_3(x, y) = 1 - 2(x - 1)/b + \sum_{p=1}^{\infty} c_p(e^{p2\pi y/b} + e^{-p2\pi c/b}\, e^{-p2\pi y/b}) \sin p2\pi(x - 1)/b. \tag{37}$$

Eq. (37) is valid only for $(x-1)$ less than $b/2$.

By considering energy storage, it is possible to write the equation for capacitance in a form analogous to (13). By substituting the potential equations (35)–(37) in this form, the relation between capacitance and the coefficients a_m, b_n, d_n, and c_p is obtained.

$$\begin{aligned} C_0/\epsilon = {} & 4K(k_0)/K(k_0') + 2c/s + 2c/b \\ & + \pi \sum_{m=1}^{\infty} (1 - e^{-m4\pi c/s}) m a_m^2 \\ & + \pi \sum_{p=1}^{\infty} (1 - e^{-p4\pi c/b}) p c_p^2 \\ & + \pi \sum_{n=1}^{\infty} (1 - e^{-n2\pi u_0})(n)(b_n^2 + d_n^2). \end{aligned} \tag{38}$$

In order to introduce the condition of continuity of potential at the matching surface, these potentials are written in terms of the coefficients:

$$\begin{aligned} \phi(u_0, v) &= 2x/s + \sum_{m=1}^{\infty} (1 + e^{-m2\pi c/s}) a_m \sin m2\pi x/s \\ &\approx v + \sum_{n=1}^{\infty} b_n \sin n\pi v \end{aligned} \tag{39}$$

$$\phi(0, v) = 1 - 2(x-1)/b + \sum_{p=1}^{\infty} (1 + e^{-p2\pi c/b})c_p \sin p2\pi(x-1)/b \approx v + \sum_{n=1}^{\infty} d_n \sin nv. \quad (40)$$

In writing (39)–(40) the following assumptions were made:

$$d_n e^{-\pi u_0} \ll b_n$$
$$b_n e^{-\pi u_0} \ll d_n. \quad (41)$$

It is now possible to relate capacitance to the potentials at the matching surfaces by determining the Fourier coefficients as functions of these potentials from (39)–(40) and substituting these expressions in (38). As in the previous example, these functions $\phi(u_0, v)$ and $\phi(0, v)$ must then be chosen to result in the maximum value of capacitance. The problem is solved approximately by assuming the form of these potential functions as

$$\phi(u_0, v) = v + b_1 \sin \pi v \approx 2x/s + a_1(1 + e^{-2\pi c/s}) \sin 2\pi x/s$$
$$\phi(0, v) = v + d_1 \sin \pi v \approx 1 - 2(x-1)/b + c_1(1 + e^{-2\pi c/b}) \sin 2\pi(x-1)/b. \quad (42)$$

Using these approximations, a_1 and c_1 can be related to b_1 and d_1:

$$a_1 = B_0 + B_1 b_1 \quad (43)$$

$$c_1 = D_0 + D_1 d_1 \quad (44)$$

$$B_0 = \frac{4}{s(1 + e^{-2\pi c/s})} \int_0^{s/2} (v - 2x/s) \sin 2\pi x/s \, dx \quad (45)$$

$$B_1 = \frac{4}{s(1 + e^{-2\pi c/s})} \int_0^{s/2} (\sin \pi v) \sin 2\pi x/s \, dx \quad (46)$$

$$D_0 = \frac{4}{b(1 + e^{-2\pi c/b})} \cdot \int_0^{b/2} [v - 2(x-1)/b] \sin 2\pi(x-1)/b \, dx \quad (47)$$

$$D_1 = \frac{4}{b(1 + e^{-2\pi c/b})} \int_0^{b/2} (\sin \pi v) \sin 2\pi(x-1)/b \, dx. \quad (48)$$

As shown in the appendix, along the matching surfaces, x and v are implicitly related by the following:

$$v = 1 - F(\theta', k_0')/K(k_0') \quad (49)$$

$$\sin \theta' = (1/k_0')\sqrt{1 - 1/t^2} \quad (50)$$

$$t = (1/k_0) \frac{t' - 1}{t' + 1} \quad (51)$$

$$t' = \frac{1 - k_0}{1 + k_0} \frac{\sinh^2 \pi x/a}{\sinh^2 \pi s/2a}. \quad (52)$$

Eqs. (43)–(52) serve to determine a_1 in terms of b_1 and to determine c_1 in terms of d_1. The integrations in (45)–(48) are carried out using numerical or graphical methods. It is now possible to express the capacitance in terms of the two parameters b_1 and d_1:

$$C_0/\epsilon = 4K(k_0)/K(k_0') + 2c/s + 2c/b + \pi(1 - e^{-4\pi c/s})(B_0 + B_1 b_1)^2 + \pi(1 - e^{-2\pi u_0})(b_1^2 + d_1^2) + \pi(1 - e^{-4\pi c/b})(D_0 + D_1 d_1)^2. \quad (53)$$

When this is maximized with respect to b_1 and then with respect to d_1, the results are

$$b_1 = \frac{-B_0}{B_1 + \dfrac{1 - e^{-2\pi u_0}}{1 - e^{-4\pi c/s}} \dfrac{1}{B_1}} \quad (54)$$

$$d_1 = \frac{-D_0}{D_1 + \dfrac{1 - e^{-2\pi u_0}}{1 - e^{-4\pi c/b}} \dfrac{1}{D_1}}. \quad (55)$$

When the last two equations are substituted in (53), the capacitance is determined. In order to present curves from which capacitance is easily obtained, it is convenient to rewrite (53) as

$$C_0/\epsilon = 4K(k_0)/K(k_0') + 2\Delta C_f'/\epsilon + 2\Delta C_f{'}_0/\epsilon. \quad (56)$$

In this, k_0 and k_0' are given by (31)–(32) and $\Delta C_f'/\epsilon$ and $\Delta C_{f0}'/\epsilon$, plotted in Figs. 6 and 7, are defined as

$$\Delta C_f'/\epsilon = c/b + (\pi/2)[(1 - e^{-2\pi u_0})d_1^2 + (1 - e^{-4\pi c/b})(D_0 + D_1 d_1)^2]$$
$$\Delta C_{f0}'/\epsilon = c/s + (\pi/2)[(1 - e^{-2\pi u_0})b_1^2 + (1 - e^{-4\pi c/s})(B_0 + B_1 b_1)^2]. \quad (57)$$

The notation used above is consistent with that used in Cohn.[4]

The first term in (56) is the capacitance of one strip to ground, operating in the odd mode, when the thickness c is zero. $\Delta C_f'/\epsilon$ is the correction for the additional fringing from the outer end of one conductor to one ground plane. $\Delta C_{f0}'/\epsilon$ is the correction for additional fringing from one half of the inner end of one conductor to the zero potential surface. Fig. 6 is a plot of the fringing capacitance correction for the outer end of the conductor. The correction is practically independent of w as long as w/b is greater than 0.2 and is independent of s as long as s/b is greater than unity. Fig. 7 shows a similar plot for the fringing capacitance correction at the inner end.

Coupled Strip Line: Even Mode

To determine the characteristic impedance for the line operating in the even mode, in which the inner conductors are maintained at equal potentials with

[4] S. B. Cohn, "Shielded coupled-strip transmission line," IRE Trans., vol. MTT-3, pp. 29–38; October, 1955.

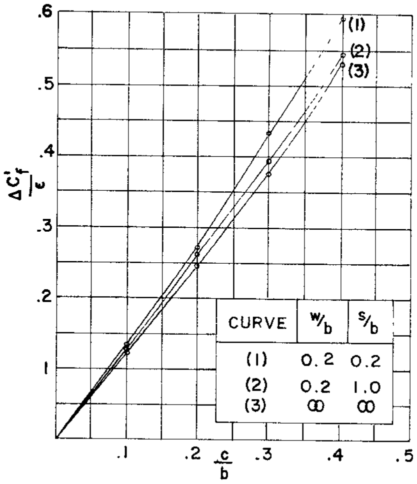

Fig. 6—Correction to outer fringing capacitance.

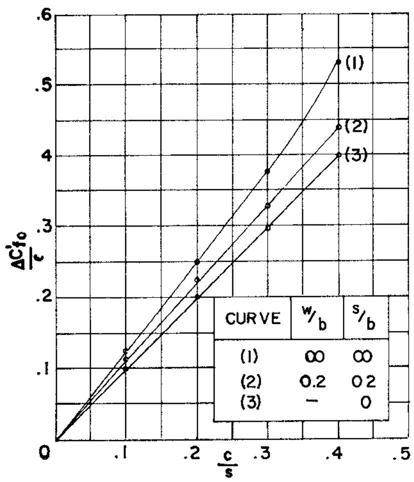

Fig. 7—Correction to inner fringing capacitance.

respect to the ground planes, attention is focused on one quadrant of the field as shown in Fig. 8. Insulating partitions are visualized as dividing the field into three regions as indicated. Following a scheme which parallels that for the odd mode, a first approximation for the capacitance can be written as

$$C_\theta/\epsilon = 4K(k_e)/K(k_e') + 2c/b, \tag{58}$$

where C_e/ϵ is the capacitance of one strip to ground and k_e and k_e' are given by

$$k_e = (\tanh \pi w/2a)(\tanh \pi(w+s)/2a) \tag{59}$$

$$k_e' = \sqrt{1 - k_\theta^2}. \tag{60}$$

With the partitions as shown, no field can exist in region 1 and therefore the corresponding capacitance term is zero.

A second approximation for the potentials can be written as

$$\phi_1(x, y) = v(0, 0) + [1 - v(0, 0)]2x/s \tag{61}$$

$$\phi_2(u, v) = v + \sum_{n=1}^{\infty} d_n(e^{-n\pi u} + e^{-2n\pi u_0}e^{n\pi u}) \sin n\pi v \tag{62}$$

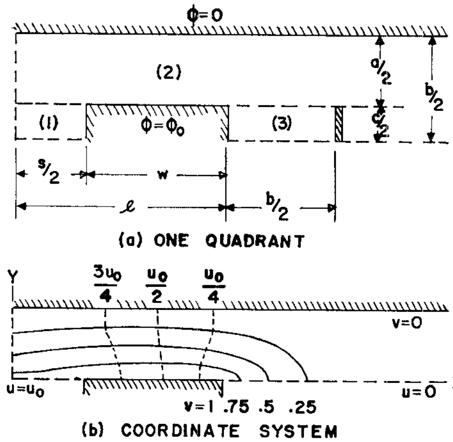

Fig. 8—Quadrant of coupled line in even mode.

$$\phi_3(x, y) = 1 - 2(x-1)/b + \sum_{p=1}^{\infty} c_p(e^{p2\pi y/b} - e^{-p2\pi y/b}e^{-p2\pi c/b}) \sin p^2\pi(x-1)/b \tag{63}$$

where $v(0, 0)$ is the value of v for $x=0$ and $y=0$. In this case the added series solution is not introduced in $\phi_1(x, y)$ since the relatively weak field in region 1 contributes little to the total capacitance. Following the procedure set out in the preceding section, an expression similar to (56) is obtained:

$$C_e/\epsilon = 4K(k_e)/K(k_e') + 2\Delta C_f'/\epsilon + 2\Delta C_{fe}'/\epsilon. \tag{64}$$

Here, k_e and k_e' are given by (59)–(60), $\Delta C_f'/\epsilon$ is given by Fig. 6 for most practical cases, and $\Delta C_{fe}'/\epsilon$ is defined as

$$\Delta C_{fe}'/\epsilon = [1 - v(0, 0)]^2(c/s). \tag{65}$$

For most proportions, this term is negligible compared with the other terms in the expression for capacitance.

Appendix

Coordinate System: Strip Line

Fig. 9 shows the transformations used in establishing the coordinate system suitable for the strip transmission line of Fig. 2. According to the Schwarz-Christoffel theorem, the variables z and t are related by

$$dz/dt = A_1(t)^{-1/2}(t-1)^{-1/2}. \tag{66}$$

This can be integrated to give

$$z = (jb/2)[1/2 - (1/\pi)\sin^{-1}(2t-1)] \tag{67}$$

where the constant of integration has been adjusted to satisfy the boundary condition at $z=0$. Again, relating w and t by means of the Schwarz-Christoffel method,

$$dw/dt = B_1(t)^{-1/2}(t-1)^{-1/2}(t-k^2)^{-1/2} \tag{68}$$

which, when integrated, yields

$$w = u_0[1 - F(\theta, k)/K(k)] + j1 \tag{69}$$

$$\sin\theta = \sqrt{t}/k. \tag{70}$$

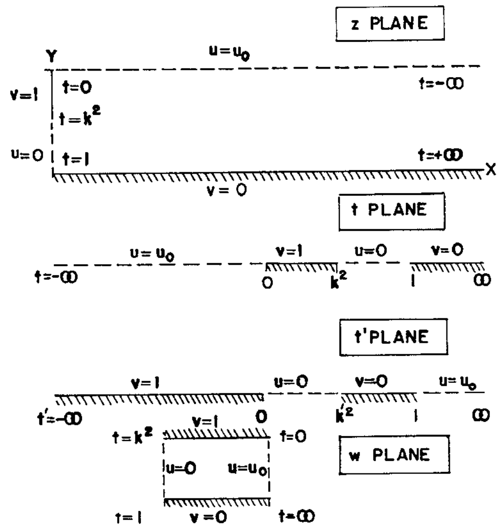

Fig. 9—Transformations for strip line coordinate system.

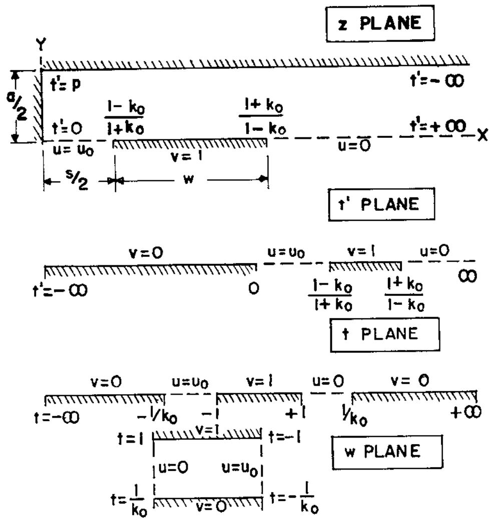

Fig. 10—Transformations for odd mode coordinate system.

Here, $F(\theta, k)$ is the incomplete elliptic integral of the first kind; $K(k)$ is the complete elliptic integral of the first kind. B_1 has been made equal to $-u_0/2K(k)$ in order to satisfy the boundary conditions at $t=0$ and $t=k^2$. Unforuntately, (69) is useful only in the range $0<t<k^2$. To obtain an expression which is useful along the matching surface, $u=0$, the transformation

$$t' = 1 - k^2/t \tag{71}$$

is utilized. Then

$$dw/dt' = (-ju_0/2K(k))(t'-1)^{-1/2}(t'-k'^2)^{-1/2}(t')^{-1/2} \tag{72}$$

$$k' = \sqrt{1-k^2}. \tag{73}$$

Eq. (72) can be integrated to give

$$w = j[1 - (u_0)F(\theta', k')/K(k)] \tag{74}$$

$$\sin\theta' = \sqrt{t'}/k'. \tag{75}$$

In order to satisfy the boundary condition at $t'=k^2$,

$$u_0 = K(k)/K(k'), \tag{76}$$

and therefore, along $u=0$,

$$v = 1 - F(\theta', k')/K(k'). \tag{77}$$

In order to relate θ' and k' directly to the z plane, (67), (71), (73), and (75) can be manipulated to give

$$\cos\theta' = (k/k')\tan\pi y/b. \tag{78}$$

Coordinate Systems: Coupled Lines

Fig. 10 shows the transformations used in establishing the coordinate system suitable for the odd mode operation of the coupled strip line shown in Fig. 5. According to the Schwarz-Christoffel transformation, z and t' are related by

$$dz/dt' = A_1(t')^{-1/2}(t'-p)^{-1/2}, \tag{79}$$

which may be integrated to give

$$z = (a/\pi)\ln\left[\sqrt{t'/-p} + \sqrt{(t'/-p)+1}\right]. \tag{80}$$

Along $y=0$, (80) can be manipulated to give the relation between t' and x:

$$t' = \frac{1-k_0}{1+k_0}\,\frac{\sinh^2 \pi x/a}{\sinh^2 \pi s/2a}. \tag{81}$$

Further, by considering the value of t' at $x=s/2$ and at $x=(w+s/2)$, the expression for k_0 can be obtained.

$$k_0 = (\tanh \pi w/2a)(\coth \pi(w+s)/2a). \tag{82}$$

Inspection of Fig. 11 shows that t and t' are related by

$$t = \frac{1}{k_0}\,\frac{t'-1}{t'+1}. \tag{83}$$

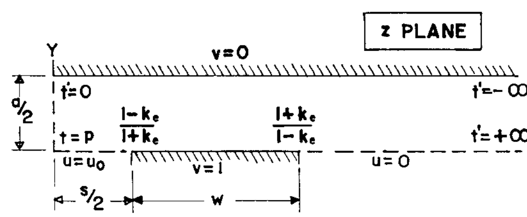

Fig. 11—Transformations for even mode coordinate system.

Now w and t can be related by means of the Schwarz-Christoffel transformation:

$$dw/dt = B_1(t^2-1)^{-1/2}(t^2-1/k^2)^{-1/2}, \tag{84}$$

which may be ingerated to give

$$w = [K(k_0)/K(k_0')][1 - F(\theta, k_0)/K(k_0)] + j1 \tag{85}$$

$$\sin\theta = t. \tag{86}$$

Eq. (85) is practically useful only in the range $-1<t<1$, that is, along $v=1$. To obtain an expression which is useful along the matching surfaces $u=0$ and $u=u_0$, (85) can be transformed to[5]

[5] P. F. Byrd and M. D. Friedmann, "Handbook of Elliptic Integrals for Engineers and Physicists," Springer Verlag, Berlin, Germany, Art. 115.02; 1954.

$$w = u + j[1 - F(\theta', k_0')/K(k_0')] \quad (87)$$

$$k_0' = \sqrt{1 - k_0^2} \quad (88)$$

$$\sin \theta' = (1/k_0')\sqrt{1 - 1/t^2}. \quad (89)$$

Eqs. (87), (89), (83), and (81) give the implicit relation between x and v along the matching surfaces.

Fig. 11 shows the z plane for the even mode operation of the coupled strip line. From it, it can be seen that the t plane, the t' plane, and the w plane sketches are identical with the odd mode sketches, except for replacing k_0 by k_e. Eqs. (89) and (86) apply, and by analogy with (85),

$$w = [K(k_e)/K(k_e')][1 - F(\theta, k_e)/K(k_e)] + j1. \quad (90)$$

Along $y=0$, by analogy with (87),

$$w = u + j[1 - F(\theta', k_e')/K(k_e')]. \quad (91)$$

Eq. (79) applies to Fig. 11 as well as to Fig. 10. When it is integrated, and boundary conditions are applied, the result is

$$z = (a/\pi) \ln [\sqrt{t'/p} + \sqrt{(t'/p) - 1}]. \quad (92)$$

Along $y=0$ this can be written as

$$t' = \frac{1 - k_e}{1 + k_e} \frac{\cosh^2 \pi x/a}{\cosh^2 \pi s/2a}. \quad (93)$$

By considering the value of t' at $x=s/2$ and at $x=(w+s/2)$ the expression for k_e can be obtained:

$$k_e = (\tanh \pi w/2a)(\tanh \pi(w + s)/2a). \quad (94)$$

Eqs. (91), (89), (83), and (93) serve to give the implicit relation between x and v along $y=0$.

ACKNOWLEDGMENT

The author wishes to express his sincere thanks to Prof. T. J. Higgins of the University of Wisconsin for his advice and encouragement during the course of this investigation.

The Impedance of a Wire Grid Parallel to a Dielectric Interface*

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Summary—**Analysis is given for the problem of reflection of a plane wave at oblique incidence on a wire grid which is parallel to a plane interface between two homogeneous dielectrics. It is assumed that the wire grid is a periodic structure and consists of thin cylindrical wires of homogeneous material. The equivalent circuit is derived where it is shown that the space on either side of the interface can be represented by a transmission line, and the grid itself is represented by a pure shunt element across one of the lines.**

INTRODUCTION

THERE HAVE been many investigations of the electromagnetic properties of thin parallel wires composed of conductive material. The first quantitative study was made by Lamb[1] in 1898 who considered the plane wave incident normally on the grid. He showed that if the diameter, $2a$, of the parallel wires was small, the reflection and transmission could be varied by changing the spacing. In 1914, von Ignatowsky[2] made a very exhaustive analysis of the scattering of incident plane waves by single metallic grids including the case where the wire spacing is comparable to the wavelength. His formulas have been reduced, extended, and applied by other authors since that time.[3–11] A very illuminating treatment has been given by MacFarlane[5] who indicated that a single grid can be represented by an impedance shunted across an infinite transmission line whose characteristic impedance is proportional to the intrinsic impedance of the

* Manuscript received by the PGMTT, June 1, 1956.
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Note: In this reference $\cos \theta \cos \phi_0 Z_i d/\eta_0$ should be replaced by $\cos^{-1} \theta \cos \phi_0 Z_i d/\eta_0$ in (25).